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An elementary proof of a formula of Jensen

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by

H.J.A. Duparc, C.G. Lekkerkerker and W. Peremans.

Jensen (Acta Math. 26 (1902), 308) stated a formula which may be brought in the following form

$$(1) \qquad \sum_{s=0}^{\infty} e^{-(a+sz)} \frac{(a+sz)^s}{s!} = \frac{1}{1-z} \text{ for } |z| \leqslant 1;\ |ze^{-z}| < \frac{1}{e},$$

which formula he proved by means of the formula of Lagrange-Burmann. In this note we give an elementary proof of (1).

We use the formula, valid for all b

$$(2) \qquad \sum_{s=0}^{r} \binom{r}{s} (-)^s (s+b)^r = (-)^r\ r! \qquad (r = 0,1,2,\ldots),$$

which may be found by applying r times the operator $z\frac{d}{dz}$ to both sides of the equality

$$\sum_{s=0}^{r} \binom{r}{s} (-)^s\ z^{s+b} = z^b(1-z)^r$$

and then taking z=1. In fact, the expression $(z\frac{d}{dz})^r\ z^b(1-z)^r$ is equal to a sum of terms each containing at least one factor 1-z, with one exception only. So if we put z=1 only one term in our sum differs from zero; its value is found by taking z=1 in $z^{b+r}(-)^r\ r!$, whence follows the result (2) .

Let f(r) be an arbitrary function of r. Then after summation over r from formula (2) it follows

$$(3) \qquad \sum_{r} \sum_{s=0}^{r} f(r)\binom{r}{s}(-)^s(s+b)^r = \sum_{r} f(r)(-)^r\ r!,$$

supposed the last sum exists. Now for f(r) we take $\frac{(-x)^r}{r!}$; then if $0 < x < 1$ formula (3) becomes

$$(4) \qquad \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{(-x)^r(-)^s(s+b)^r}{s!(r-s)!} = \sum_{r=0}^{\infty} x^r = \frac{1}{1-x}.$$

If further we suppose $xe^x < \frac{1}{e}$, we have

$$(5) \qquad \sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{(-x)^r(-)^s(s+b)^r}{s!(r-s)!} = \sum_{s=0}^{\infty} \frac{x^s(s+b)^s}{s!} \sum_{r=s}^{\infty} \frac{(-x(s+b))^{r-s}}{(r-s)!},$$

because the summations may be interchanged. In fact the double series on the right hand side converges absolutely, for

$$\sum_{s=0}^{\infty} \frac{x^s|s+b|^s}{s!} \sum_{r=s}^{\infty} \frac{(x|s+b|)^{r-s}}{(r-s)!} = \sum_{s=0}^{\infty} \frac{x^s|s+b|^s}{s!}\ e^{x|s+b|} < \infty$$

on account of

$$x\left|\frac{s+b+1}{s+1}\right| e^x \left|1+\frac{1}{s+b}\right|^s \to xe^{x+1} < 1.$$

Then from (4) and (5) it follows putting $r-s = t$

$$\frac{1}{1-x} = \sum_{s=0}^{\infty} \frac{x^s(s+b)^s}{s!} \sum_{t=0}^{\infty} \frac{(-x(s+b))^t}{t!} = \sum_{s=0}^{\infty} \frac{x^s(s+b)^s}{s!} e^{-x(s+b)}.$$

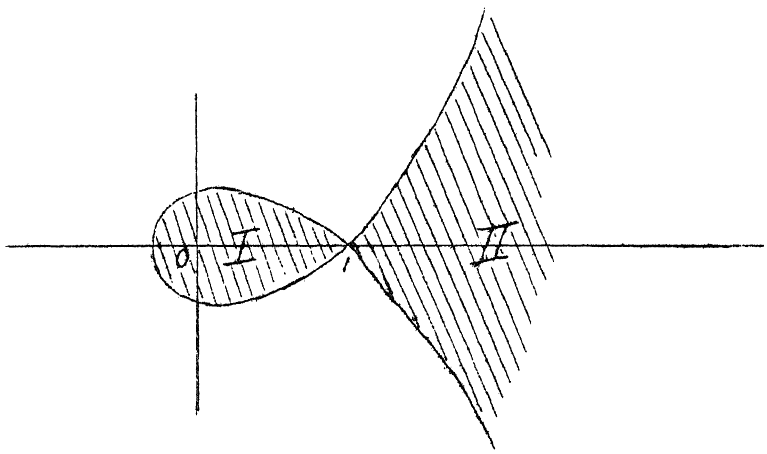
If in this result we put $bx = a$, we get for $0 < xe^x < 1$ Jensen's formula

$$(6) \qquad \sum_{s=0}^{\infty} e^{-(a+xs)} \frac{(a+xs)^s}{s!} = \frac{1}{1-x}.$$

Now the series

$$(7) \qquad \sum_{s=0}^{\infty} e^{-(a+zs)} \frac{(a+zs)^s}{s!}$$

converges in the domain $|ze^{-z}| < \frac{1}{e}$, drawn below.



Since for $0 < xe^x < 1$ the formula (6) holds, by analytic continuation, we find that for all z in the convex domain I, determined by

$$|ze^{-z}| < \frac{1}{e}; \quad |z| < 1,$$

the series (7) converges to $\frac{1}{1-z}$. The series (7) is also convergent in the domain II, determined by

$$|ze^{-z}| < \frac{1}{e}; \quad |z| > 1,$$

although the value can not be obtained by analytic continuation from domain I. Its sum however then can be found by the following consideration. First remark that if z lies in domain I we have proved

$$\sum_{s=0}^{\infty} \frac{(ae^{-z}+sze^{-z})^s}{s!} = \frac{e^a}{1-z},$$

hence putting $ae^{-z} = c$

$$(8) \qquad \sum_{s=0}^{\infty} \frac{(c+sze^{-z})^s}{s!} = \frac{e^{ce^z}}{1-z}.$$

Now suppose that z lies in the domain II. Then there exists exactly one point ζ in domain I satisfying

(9) $\qquad ze^{-z} = \zeta e^{-\zeta}$.

To prove this assertion we put $ze^{-z} = \alpha$, hence $|\alpha| < \frac{1}{e}$. Denote the number of points ζ in the domain I, satisfying $f(\zeta) = \zeta e^{-\zeta} - \alpha = 0$, by $N(\alpha)$. Then we have taking the integral in positive sense along the boundary C of domain I

$$N(\alpha) = \frac{1}{2\pi i}\int_C \frac{f'(w)}{f(w)}\,dw.$$

This function $N(\alpha)$ is for $|\alpha| < \frac{1}{e}$ continuous and has for each such α an integral value, hence this function is a constant. Since obviously $N(0) = 1$, we have $N(\alpha) = 1$ for $|\alpha| < \frac{1}{e}$.

Introducing the solution ζ of equation (9) the relation (8) becomes

$$\sum_{s=0}^{\infty} \frac{(c+sze^{-z})^s}{s!} = \sum_{s=0}^{\infty} \frac{(c+s\zeta e^{-\zeta})^s}{s!} = \frac{e^{ce^{\zeta}}}{1-\zeta},$$

hence using $ae^{-z} = c$

$$\sum_{s=0}^{\infty} \frac{e^{-zs-a}(a+sz)^s}{s!} = \frac{e^{ce^{\zeta}-a}}{1-\zeta} = \frac{e^{ae^{\zeta-z}-a}}{1-\zeta} = \frac{e^{a\frac{\zeta}{z}-a}}{1-\zeta}.$$

So we found for $|ze^{-z}| < \frac{1}{e}$ the following result

$$\sum_{s=0}^{\infty} e^{-a-sz}\,\frac{(a+sz)^s}{s!} = \begin{cases} \dfrac{1}{1-z} & \text{if } |z| < 1, \\ \dfrac{e^{a\frac{\zeta}{z}-a}}{1-\zeta} & \text{if } |z| > 1, \text{ where } \zeta \text{ is determined by (9) and } |\zeta| < 1. \end{cases}$$

We give some other applications of formula (3). If we suppose b positive integral and if we take

$$f(r) = \frac{b!}{r!r!(b-r)!},$$

then (3) gives

$$\sum_{r=0}^{b}\sum_{s=0}^{r} \binom{b}{r}\binom{r}{s} \frac{(-)^s(s+b)^r}{r!} = \sum_{r=0}^{b} \frac{b!(-)^r}{r!(b-r)!} = 0.$$

If we however take

$$f(r) = \frac{(-)^r b!}{r!r!(b-r)!},$$

we get from (3)

$$\sum_{r=0}^{b}\sum_{s=0}^{r} \binom{b}{r}\binom{r}{s} \frac{(-)^{s+r}\,(s+b)^r}{r!} = \sum_{r=0}^{b} \frac{b!}{r!(b-r)!} = 2^b,$$

hence

$$\sum_{b=0}^{\infty}\sum_{r=0}^{b}\sum_{s=0}^{r} \frac{(-)^{s+r}(s+b)^r}{s!r!(b-r)!(r-s)!} = \sum_{b=0}^{\infty} \frac{2^b}{b!} = e^2.$$

If in (2) both members are multiplied by $\frac{(-)^r}{r!b^r}$, we obtain after summation over b

$$\sum_{b=1}^{\infty}\sum_{s=0}^{r} \frac{(1+\frac{s}{b})^r(-)^{s+r}}{s!(r-s)!} = \sum_{b=1}^{\infty} \frac{1}{b^r} = \zeta(r).$$

If in (3) we take $f(r) = \frac{(-z)^r}{r!r!}$,we get

$$\sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{(-z)^r(-)^s(s+b)^r}{r!s!(r-s)!} = \sum_{r=0}^{\infty} \frac{z^r}{r!} = e^z.$$

Now the series in the left hand side of this relation converges absolutely because

$$\sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{|z(s+b)|^r}{r!s!(r-s)!} < \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{r=0}^{\infty} \frac{|z(s+b)|^r}{r!} =$$

$$= \sum_{s=0}^{\infty} \frac{e^{|zs+zb|}}{s!} \leqslant e^{|zb|} \sum_{s=0}^{\infty} \frac{e^{|z|s}}{s!} = e^{|zb|+e^{|z|}}.$$

Hence putting $r = s+t$ and changing the order of summation we find

$$\sum_{r=0}^{\infty} \sum_{s=0}^{r} \frac{(-z)^r(-)^s(s+b)^r}{r!s!(r-s)!} = \sum_{s=0}^{\infty} \frac{(zs+zb)^s}{s!} \sum_{t=0}^{\infty} \frac{(-zs-zb)^t}{t!(t+s)!} =$$

$$= \sum_{s=0}^{\infty} \frac{(zs+zb)^{\frac{s}{2}}}{s!} J_s(2\sqrt{zs+zb}),$$

so

$$\sum_{s=0}^{\infty} \frac{(zs+zb)^{\frac{s}{2}}}{s!} J_s(2\sqrt{zs+zb}) = e^z.$$